

001 17 1951

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1315

DISPLACEMENT EFFECT OF THE LAMINAR BOUNDARY
LAYER AND THE PRESSURE DRAG

By H. Görtler

Translation of "Verdrängungswirkung der laminaren Grenzschichten
und Druckwiderstand." Ingenieur-Archiv, vol. 14, 1944



Washington

October 1951

NACA LIBRARY
LANGLEY AERONAUTICAL LABORATORY
Langley Field, Va.



3 1176 01441 2002

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1315

DISPLACEMENT EFFECT OF THE LAMINAR BOUNDARY

LAYER AND THE PRESSURE DRAG*

By H. Görtler

1. INTRODUCTION

The following considerations refer to two-dimensional, not necessarily steady laminar movements of an infinitely extended fluid without free surfaces, for very small friction, about a body¹ immersed in this fluid; they deal with the mutual influence between the boundary layer developing in the proximity of the body and the outer flow which is practically free from friction.

We denote by X and Y the coordinates of a Cartesian right-hand system of the plane fixed with respect to the body, by x and y the customary boundary layer coordinates defined in a zone near the wall (that is, an orthogonal right-hand system where x signifies the wall-arc length measured from a fixed contour point, y the wall distance measured positively from the wall toward the fluid).

Furthermore, $R(x)$ is assumed to be the radius of curvature of the wall which we define in the manner that has become customary for boundary layer investigations, in contrast to the usual mathematical definition, as positive for walls which are convex with respect to the fluid.² Finally, u and v are assumed to be the velocity components of the boundary-layer flow, U and V the corresponding components of the potential flow about the body in x or y direction, \bar{U} and \bar{V} the velocity components of the potential flow in X or Y direction.

*"Verdrängungswirkung der laminaren Grenzschichten und Druckwiderstand." Ingenieur-Archiv, vol. 14, 1944, pp. 286-305.

¹The contour of the cylinder cross section in the flow plane should, if no other requirements are made explicitly, be of continuous curvature and free of multiple point singularities; it may go to infinity.

²In order to have in the defined range of x , y a reversible one-to-one relation between X , Y and x , y , the requirement must be made that there $R + y \neq 0$. This does not impose any further limitation, however, since according to presupposition in the boundary layer region $|y/R| \ll 1$.

If the well-known boundary layer omissions are permissible, the general hydrodynamic equations of motion are reduced to the following boundary-layer equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (1.1a)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.1b)$$

where t signifies the time, ρ the constant density, and ν the constant kinematic viscosity of the fluid. Furthermore, $p = p(x, t)$ is the pressure impressed on the boundary layer by the outer flow. Insofar as the outer flow noticeably deviates from the regular potential flow about the body, $p(x, t)$ must be determined on the basis of special considerations or be taken from pressure measurements. In the present report, we consider only cases where the potential-theoretical pressure distribution in first approximation is sufficient for the boundary-layer calculation according to equations (1.1). In this boundary-layer theory approximation, the potential velocity prevailing at the "outer edge" of the boundary layer is replaced by the potential flow prevailing about the body itself, thus by $U(x, 0, t)$, and the pressure term in equation (1.1a) is calculated from Euler's equation of motion with $U(x, 0, t) \equiv U_0(x, t)$ as

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x}. \quad (1.2)$$

The boundary condition required in addition to the adherence conditions $u = v = 0$ at the wall $y = 0$ is that U assume asymptotically for increasing $y\sqrt{Re}$ the value³ $U_0(x, t)$. (Re is the Reynolds number formed with a characteristic length and a characteristic velocity.)

The described omissions, correct in the limit for indefinitely increasing Reynolds numbers, are only an approximation for the large but still finite Re -values for which laminar flow is plainly still possible. Particularly the connection between the boundary-layer flow thus calculated and the outer potential flow, assumed to be undisturbed,

³The imaginary line, denoted above, in the customary manner of expression, as the "outer edge" of the boundary layer, runs at the distance from the wall $y = \delta(x, t)$ ("boundary-layer thickness") at which u practically attains the value $U_0(x, t)$ for the accuracy requirements in each case.

does not satisfy continuity. Thus the question arises: What actual course does the outer, practically frictionless flow which is modified by the boundary layer take? This problem and the related one regarding the calculation of the pressure drag will be discussed below. The considerations will be applied to an example in all details.

2. DISPLACEMENT EFFECT OF THE BOUNDARY LAYER ON THE OUTER FLOW

We assume a body which, from a state of rest, is set in motion relative to the surrounding fluid. In this case, the boundary layer gradually developing from the start of the movement will be very thin at first, so that for sufficiently small times of movement the regular potential flow about the body itself, as outer flow, may be taken as a basis for the boundary layer calculation in good approximation. The well-known investigation concerning the origin of the boundary layers (H. Blasius⁴), and later extensions^{5,6} are based on this assumption. However, with increasing times t the boundary layer increases, for a finite Reynolds number, up to a noticeable thickness; in general, it will influence the outer flow by mass displacement away from the wall to a corresponding extent. For a corresponding improvement of the customary boundary-layer theory approximation sketched in section 1, one must therefore consider the mutual influence of the increasing boundary layer and the outer-flow. It suggests itself to attain such an improvement in first approximation by using at first the displacement effect on the basis of the original boundary-layer calculation, and hence determining a correspondingly corrected course of the outer flow. In a second step one would then have to calculate the boundary-layer flow anew, taking the above new outer flow as a basis. However, it will generally not be possible to carry out this improved boundary-layer calculation, using the new outer pressure distribution, according to the old scheme (equations (1.1) with the boundary conditions formulated there in the text), since the curvature effects, so far neglected in the boundary-layer equations, generally contribute amounts which can no longer be neglected within the scope of such an improved boundary-layer calculation. We shall return to this later on (section 4).

⁴Blasius, H.: Grenzsichten in Flüssigkeiten mit kleiner Reibung (The Boundary Layers in Fluids with Little Friction). Dissertation Göttingen, Zeitschrift für Mathematik und Physik 56, 1908, p. 1. (Available as NACA TM 1256.)

⁵Boltze, E.: Grenzsichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung. (Boundary Layers on Bodies of Revolution in Fluids with Little Friction). Diss. Göttingen, 1908.

⁶Goldstein, S., and Rosenhead, L.: Proc. Cambridge Phil. Soc. 32, 1936, p. 392.

The improved flow calculation aspired to is of decisive importance for the numerical determination of the pressure drag⁷. This becomes particularly obvious, if one visualizes a body of finite dimension transferred from a state of rest into a state of rectilinear-uniform movement relative to the surrounding fluid of infinite extension. For this terminal state the potential-theoretical pressure distribution, so far, in first approximation, taken as a basis for the boundary-layer calculation, does not yield any pressure drag (d'Alembert's paradox). Thus, small as the changes in pressure gradient at the body caused here by the boundary layer may be relative to the potential-theoretical pressure gradient, they alone constitute the pressure drag.

We want to emphasize here once more that, as we discussed before in the introduction, we do not include in our considerations flows where, for instance, by pronounced separation of the boundary layer, the outer frictionless flow is considerably transformed compared to the potential flow about the body. Of course, precisely such processes are of foremost importance for the originating pressure drag in many flow problems. Such processes cannot be included within the scope of the following considerations which are based on the boundary-layer theoretical approximation. However, we should like to refer here to a related report by M. Schwabe⁸ where the pressure distribution after completed boundary-layer separation is determined according to an empirical formulation for the example of the circular cylinder set into rectilinear-uniform motion from a state of rest. The space taken up by the pair of vortices developing on the back of the cylinder after the separation is determined by observation and then simulated by calculation by superposition of a suitable time-dependent source-sink flow on the ordinary potential flow about the cylinder. One then obtains outside of this space a streamline pattern which corresponds well to actual conditions, and a pressure drag caused by the nonsteady acceleration fields.

Furthermore, we want to point out that J. Pretsch⁹ developed an approximative method for theoretical determination of the pressure drag

⁷"pressure drag" = the component in direction of the movement of the resultant pressure force on the entire body surface, also called (less correctly) "form drag". Pressure drag + friction drag = total drag (presupposing non-existence of free surfaces).

⁸Schwabe, M.: Über Druckermittlung in der nichtstationären ebenen Strömung (On the Determination of Pressure in a Nonsteady Two-Dimensional Flow). Diss. Göttingen. Ing.-Archiv 6, 1935, p. 34.

⁹Pretsch, J.: Zur theoretischen Berechnung des Profilwiderstandes (Concerning Theoretical Calculation of the Profile Drag). Diss. Göttingen, Jahrbuch 1938 der deutschen Luftfahrtforschung, p. I 60. (Available as NACA TM 1009.)

of profiles in a steady flow by taking the displacement effect of the boundary layer into account. In a procedure similar to ours he presupposes that no separation of the boundary layer, or only a slight one, takes place; in the steady case this signifies, however, a limitation to slender profiles at a small angle of attack so that here the total profile drag is due in the greatest part to surface friction. Our consideration, related in the basic idea, refers at the outset to non-steady flows as well and aims chiefly at the problem of the drag origin in movements of cylindrical bodies of arbitrary profile from a state of rest for small times after the start of motion, naturally without being limited to these movements. Also, it follows a different method. Whereas Pretsch makes the additional assumption that the friction losses in the wake behind the body may be neglected, probably satisfied in good approximation in view of his presupposition quoted above ("Bodies of Small Drag"), and then is able to determine, within this scope, the pressure drag from momentum considerations in a simple and general manner without actually having to calculate the corrected outer potential flow in every single case, we abstain from this or a similar simplifying assumption because we set ourselves a problem of a different type.

In the following considerations we shall first deal with the first step of problem formulation, namely, the correction of the outer potential flow by consideration of the displacement effect (resulting from the boundary-layer flow calculated in first approximation), and shall discuss a few questions arising in case of nonsteady conditions. After that the second step which leads to the calculation of the pressure drag will be treated in general and carried out numerically on an example.

3. CORRECTION OF THE OUTER POTENTIAL FLOW

First, one has to find an appropriate measure for the displacement effect of the boundary layer on the outer flow. For that purpose the well-known boundary-layer theoretical length presents itself which is called "displacement thickness" and denoted by δ^* .

The formulation of the boundary condition for $u(x,y,t)$ for indefinitely increasing $y\sqrt{Re}$ was based on the conception that the potential theoretical velocity distribution for the large (though still finite) values of the Reynolds numbers of interest may be regarded practically as constant in first approximation for the very small thickness of the boundary layer and may, therefore, be put equal to $U(x,0,t)$. The definition of the displacement thickness δ^* is likewise based on the conception of this streamline approximation $[U(x,y,t) = U_0(x,t)]$ and thus for reasons of continuity $V(x,y,t) = -y\partial U_0/\partial x]$.

The quantity δ^* is explained as follows: The fluid volume transported at a fixed point x at a fixed time t between the wall $y = 0$ and the outer edge $y = \delta(x, t)$ of the boundary layer in unit time

equals $\int_0^{\delta} u \, dy$. The potential flow about the body would transport, if the streamline approximation described above were taken for a basis, the volume $U_0(x, t)\delta$ through the same cross-section in unit time. The difference between these two quantities, that is, the loss in rate of flow per unit time caused by the viscosity effect, leads by virtue of

$$U_0\delta - \int_0^{\delta} u \, dy = U_0\delta^*$$

to the definition of the length δ^*

$$\delta^*(x, t) = \frac{1}{U_0} \int_0^{\delta} (U_0 - u) \, dy. \quad (3.1)$$

[Compare figure 1; the hatched areas are equal. Of course, any other length $y = y_1 > \delta$ may be selected instead of the upper limit δ of the integral in equation (3.1), as long as the velocity $U(x, y, t)$ in $0 \leq y \leq y_1$ is replaced by $U_0(x, t)$.] According to definition, δ^* is, therefore, a measure of the displacement of the streamlines of the outer potential flow away from the body on the basis of the reduction (caused by the friction layer near the body) in the quantity of fluid flowing by the point x at the time t . (Therein a streamline in its identity for all times is prescribed by the fact that it, together with the wall $y = 0$, includes a stream tube of temporally constant through flow.)

In order to obtain, instead of the potential flow about the prescribed body, a corrected outer potential flow which takes the displacement effect of the boundary layer into account, we visualize the following model flow: Outside of a line $y = \delta^*(x, t)$ a potential flow with $y = \delta^*$ as streamline is assumed to flow. Within $0 \leq y < \delta^*$ one assumes water at rest relative to the body. In this model flow the same quantity of fluid is to flow past the body per unit time as in the viscous flow. Equation (3.1) then indicates the value of $\delta^*(x, t)$ in first approximation for sufficiently high Re values.

As to the line $y = \delta^*$ which is to be a streamline of our substitute flow, it must be taken into consideration that for the general nonsteady case the course of this line varies with time. The line $y = \delta^*$ then is a movable dividing line between potential flow and water at rest. In contrast to the dynamic dividing lines, it therefore consists in general not permanently of the same fluid particles, but represents, for reasons of continuity, a permeable line. In mathematical formulation the boundary condition to be stipulated expresses that the normal component of the potential-flow velocity along the dividing line $y = \delta^*(x, t)$ should vanish every moment. Then $y = \delta^*$ is a streamline.

A simple example which we shall investigate more thoroughly later (section 5) will serve to clear up these conditions. We assume that a circular cylinder of the radius R is set at the time $t = 0$, from a state of rest, relative to the surrounding infinitely extended fluid, into a rectilinear and uniformly accelerated motion perpendicular to its axis. The frictionless flow relative to the body is then given by the velocity potential

$$\phi_0 = bt \left(r + \frac{R^2}{r} \right) \cos \vartheta$$

where r and ϑ signify polar coordinates in the flow plane, referred to the center of the circle, $r = R + y$, $\vartheta = \pi - x/R$, and b denotes a constant acceleration. This potential flow yields the pressure variation on $r = R$ required for the usual boundary-layer calculation. For very small times (small compared to the time $t = t_A$ of the start of separation) this calculation results in a displacement thickness δ^* increasing proportionally to \sqrt{vt} . For these very small times after the start of the motion the potential of our improved outer flow therefore reads

$$\phi_1 = bt \left[r + \frac{(R + c\sqrt{vt})^2}{r} \right] \cos \vartheta$$

$$(t \geq 0, r \geq R + c\sqrt{vt},$$

$$\delta^* = c\sqrt{vt}, c = \text{const.})$$

The line $y = c\sqrt{vt}$ is a streamline. While it travels out into the fluid starting from $t = 0$, the streamline pattern of the entire outer flow correspondingly varies continuously¹⁰.

In order to clarify the conception we shall add a few remarks regarding the mass displacement of the boundary layers. The streamline displacement thickness δ^* does by no means always represent at the same time a measure for the mass displacement. This becomes immediately clear from the following simple example. An unlimited plane wall which up to the time $t = 0$ is at rest relative to the surrounding infinitely extended fluid is assumed to be moved in itself, according to an arbitrary acceleration law, starting from $t = 0$. A certain velocity profile (which is known to be easily defined) develops, and one obtains a displacement thickness δ^* different from zero. However, a mass displacement away from the wall does not take place due to $v \equiv 0$ [$u = u(y, t)$]. We could have made clear the difference in the two displacement phenomena in a perfectly analogous manner on the case of the circular cylinder set rotating about its axis from a state of rest; thus we should have spared ourselves the fiction of an infinitely extended body. Here, just as in the above limiting case of the plane wall, a boundary layer develops; a mass displacement away from the cylinder in radial direction, however, cannot take place due to reasons of continuity and symmetry. If one relinquishes this symmetry by selecting for instance instead of the circular cylinder a cylinder slightly wavy in comparison, the mass displacement normal to the contour will be, in general, different from zero; however, the length δ^* does not present a measure for it.

¹⁰We shall give here for comparison the velocity potential $\hat{\phi}$ of the corresponding flow about an expanding impermeable circular cylinder. If its contour is prescribed by $r = R(t)$, one has

$$\hat{\phi} = bt \left[r + \frac{R^2(t)}{r} \right] \cos \vartheta + R(t) \frac{dR(t)}{dt} \ln \frac{r}{R}$$

The additively appearing source yields that additional fluid in the outer space $r = R(t)$ which the impermeable circular cylinder itself displaces while expanding

$$\left(\frac{d\hat{\phi}}{dr} \right)_{r=R(t)} = \frac{dR(t)}{dt}$$

Looking for a measure for the mass displacement, one may start from the relation following from equation (3.1) by differentiation with respect to x and consideration of the continuity equation

$$\frac{\partial}{\partial x} (U_0 \delta^*) = v(x, \delta, t) + \delta \frac{\partial U_0}{\partial x} (x, t) \quad (3.2)$$

(Again any $y = y_1 > \delta$ may be substituted for δ). The right side indicates the excess of the v -velocity at the edge of the boundary layer compared to the V -velocity $-\delta \partial U_0 / \partial x$ of the corresponding frictionless flow prevailing there. Thus the entire volume displaced by friction effect between two points x_0 and x_1 per unit time is at any rate always prescribed by

$$\int_{x_0}^{x_1} (v - V) dx = [U_0 \delta^*]_{x=x_1} - [U_0 \delta^*]_{x=x_0}$$

We now visualize, as in the definition of δ^* , a model flow. We assume a frictionless potential flow, with the quantity δ_* generally different from δ^* , to be prevailing for $y \geq \delta_*(x, t)$; the length δ_* is assumed to be fixed so that precisely the fluid which actually is displaced by the friction effect would flow through $0 \leq y \leq \delta_*$ with the velocity U_0 . Inside $0 \leq y \leq \delta_*$ we assume for our model flow water at rest relative to the body. Then there follows from equation (3.2)

$$\frac{\partial}{\partial x} (U_0 \delta_*) = \frac{\partial}{\partial x} (U_0 \delta^*). \quad (3.3)$$

Hence there results by integration for the "mass displacement thickness" δ_* the statement

$$U_0 (\delta_* - \delta^*) = f(t), \quad (3.4)$$

that is, this difference is only dependent on t .

Hence one can recognize: If one were to proceed in the determination of the first correction of the outer potential flow described above,

and therewith of the pressure field correction for large Re -values, in such a manner that one would make not $y = \delta^*$ but $y = \delta_*$ for all times the streamline of the substitute flow, the result would remain unchanged, according to equation (3.4). Both models differ at any moment only in the numbering of the streamlines.

As to the relation between δ^* and δ_* , one may state: If one assumes that at the respective time t the stream tube which includes at the point x the region $0 \leq y \leq y_1$ with $y_1 \geq \delta$ followed up upstream finally blends completely in a region where the flow is frictionless (case of approach flow), $\delta^* \equiv \delta_*$. If, however, this presupposition is not fulfilled, as for instance in the examples with vanishing δ_* selected above, the conclusion drawn above is not valid, either. In every flow of this type the streamtube just considered is either bounded by the wall in its entire course upstream, or it leaves the wall in stretches as when a separation of the boundary layer material from the body with subsequent readherence takes place¹¹ upstream (case of longitudinal flow.) The value of δ_* at the time t can be given for all x if one is able to give, in addition to the variation of $\delta^*(x,t)$, known according to equation (3.1), the value of δ_* at a single point $x = x_0$;¹² for equation (3.4) then yields only the difference statement

$$\left[U_0 \delta_* \right]_{x_0}^x = \left[U_0 \delta^* \right]_{x_0}^x$$

¹¹It is always presupposed that the flow in the x -interval of interest of the body contour and outside of the boundary layer next to the body is frictionless. Thus for instance the case where the body gets into its own wake during the motion is left out of consideration.

¹²It must be noted, though, that cases exist where it is impossible on principle to give such a value. Let us visualize for instance the boundary-layer flow originating when an unlimited wall, deviating from a mean plane by surface waviness, is moved in its mean plane out of a state of rest. If no special conditions prevail, the expression for the total volume displaced at the time t up to a point x in unit time

$$\int_{-\infty}^x \left[v(x, y_1, t) + y_1 \frac{\partial U_0}{\partial x}(x, t) \right] dx$$

will not even be unique.

Therewith we shall close this little digression concerning mass displacement. Following we shall make use only of the streamline displacement thickness δ^* .

4. CALCULATION OF THE PRESSURE DRAG IN FIRST APPROXIMATION

With the improvement in the calculation of the outer potential flow a more accurate specification of the pressure distribution in the outer frictionless flow becomes possible. However, we must warn here against the following fallacy: In general, one cannot use the resulting pressure distribution at the edge of the boundary layer as impressed pressure $p(x,t)$ for an improved boundary-layer calculation according to the equations (1.1) and (1.2), for the obtained correction of the outer pressure gradient is, in general, of the same order of magnitude as the terms which have been neglected in the equations of motion. To obtain an improved boundary-layer calculation it would thus be necessary to take corresponding further terms into consideration in the equations of motion as well.

The general Navier-Stokes equations of motion and the continuity equation read in our curvilinear coordinates x, y in full strictness¹³

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + \frac{R}{R+y} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{uv}{R+y} &= - \frac{1}{\rho} \frac{R}{R+y} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial y^2} + \right. \\ &\quad \frac{1}{R+y} \frac{\partial u}{\partial y} + \frac{R^2}{(R+y)^2} \frac{\partial^2 u}{\partial x^2} - \\ &\quad \frac{u}{(R+y)^2} + \frac{2R}{(R+y)^2} \frac{\partial v}{\partial x} - \\ &\quad \frac{R}{(R+y)^3} \frac{dR}{dx} v + \\ &\quad \left. \frac{Ry}{(R+y)^3} \frac{dR}{dx} \frac{\partial u}{\partial x} \right] \end{aligned} \right\} \quad (4.1a)$$

¹³Compare, for instance: Tollmien, W.: Grenzsichttheorie (Boundary-Layer Theory). Handbuch der Experimentalphysik (Manual of Experimental Physics) Vol. 4, Part I, p. 248, Leipzig 1931.

$$\begin{aligned}
\frac{\partial v}{\partial t} + \frac{R}{R+y} u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{u^2}{R+y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + v \left[\frac{\partial^2 v}{\partial y^2} - \frac{2R}{(R+y)^2} \frac{\partial u}{\partial x} + \right. \\
\left. \frac{1}{R+y} \frac{\partial v}{\partial y} + \frac{R^2}{(R+y)^2} \frac{\partial^2 v}{\partial x^2} - \right. \\
\left. \frac{v}{(R+y)^2} + \frac{R}{(R+y)^3} \frac{dR}{dx} u + \right. \\
\left. \frac{Ry}{(R+y)^3} \frac{dR}{dx} \frac{\partial v}{\partial x} \right],
\end{aligned}
\tag{4.2a}$$

$$\frac{R}{R+y} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{R+y} = 0.
\tag{4.3a}$$

Therein $R(x)$ denotes the radius of curvature of the wall contour $y = 0$ (contrary to the mathematical definition positive on walls convex with respect to the flow).

If one considers in these equations, on the basis of the customary estimates founded on the physical picture, only the terms of highest order for very large Reynolds numbers $Re = UL/\nu$ (U is a characteristic velocity, L a characteristic length), one arrives, in the known manner, at the boundary-layer equations (1.1)¹⁴. If one considers in improved approximation also the terms of the order $O(\delta/L)$ compared with $O(1)$, there results

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} - v \frac{\partial^2 u}{\partial y^2} = \frac{y}{R} u \frac{\partial u}{\partial x} - \frac{uv}{R} + \frac{y}{\rho R} \frac{\partial p}{\partial x} + \frac{v}{R} \frac{\partial u}{\partial y},
\tag{4.1}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{u^2}{R},
\tag{4.2}$$

¹⁴Compare, for instance, W. Tollmien, footnote 13.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{v}{R} \frac{\partial u}{\partial x} - \frac{v}{R} \quad (4.3)$$

On the left side are the terms from equation (1.1), on the right the newly added terms.

The correction of the outer pressure we obtained above is, in general, of this same order $O(\delta/L)$. Thus, if one wants to perform with this improved outer flow an improved boundary-layer calculation (second approximation for large Re), one can in general no longer neglect the right sides and can, therefore, no longer calculate with a pressure $p = p(x, t)$ impressed on the boundary layer. (Compare additional remark at the end of this section.)

For calculation of the pressure drag in first approximation for large Re , the problem that interests us here, the solution is simpler. We now know, according to the expositions of section 3, the outer pressure field, far remote from the body, sufficiently accurately; it only remains for us to continue this pressure field up to the body surface by determining the pressure gradient in y -direction through the boundary layer. Equation (4.2) serves this purpose: It yields $\partial p / \partial y$ sufficiently accurately for this approximation, if we substitute in it on the right side u from the first boundary-layer approximation.

If $y = \delta(x, t)$ denotes the "outer edge" of the boundary layer, and $p[x, \delta(x, t), t]$ is the pressure distribution of the improved outer potential flow along this line, one has for $0 \leq y \leq \delta$ in this approximation

$$p(x, y, t) = p(x, \delta, t) - \frac{\rho}{R(x)} \int_y^\delta u^2 dy. \quad (4.4)$$

The pressure drag of the unit length of the cylinder in the flow becomes

$$\left. \begin{aligned} W_D &= \int_K p(x, 0, t) \cos \varphi dx \\ &= \int_K \left[p(x, \delta, t) - \frac{\rho}{R(x)} \int_0^\delta u^2 dy \right] \cos \varphi dx, \end{aligned} \right\} \quad (4.5)$$

Wherein φ signifies the angle between surface normal and main flow direction, and the integral is to be formed over the entire contour K of the cylinder cross section.¹⁵

¹⁵Additional remark at the time of proof correction: Regarding the problem of an improved calculation of the boundary layer flow (second approximation for moderately large Reynolds numbers), not followed up further in the present report, the following calculation procedure seems to me to be promising:

1. Calculation of the boundary layer in the customary first approximation for very large Re .

2. Hence correction of the outer frictionless flow according to Ziff's method.

3. Improved calculation of the pressure field $p(x,y,t)$ in the boundary-layer zone according to equation (4.4).

4. Again calculation of the velocity components u and v of the boundary layer in second approximation from equations (4.1) and (4.3), using the pressure field calculated above and replacing the right sides of equations (4.1) and (4.3) by the known expressions of first approximation so that the newly added terms of the order $O(\delta/R)$ compared with 1 appear in the calculation as prescribed functions.

5. EXAMPLE¹⁶

Following we shall consider as an example a nonsteady flow which originates if a body is, from a state of rest, set into a rectilinear motion relative to the surrounding infinitely extended fluid, onward from $t = 0$. The relative velocity of the undisturbed approach flow with respect to the body visualized as being at rest is assumed to

¹⁶The simplest, almost trivial example on which the method developed may be tested is the case of the plane steady stagnation point flow. Here the strict solution of the Navier-Stokes equations is known, and the flow near the wall calculated on the basis of the boundary-layer theory is known to agree with the exact solution. If one replaces the potential-theoretical stagnation point flow at the wall by the stagnation point flow at the wall shifted by $\delta^*(= \text{const.})$, one obtains as the corrected outer flow for $y > \delta$ also full agreement with the exact solution. Since the outer pressure gradient parallel to the wall has remained unchanged in this correction and the wall is plane, an improvement of the boundary-layer calculation proves to be impossible as it has to be. Thus the exact solution has already been attained with this one step.

A further example with plane body boundaries (for which the boundary-layer equations (1.1) therefore are valid except for terms of the order $O(\delta^2/R^2)$ compared with 1) is the case of longitudinal flow over a plate. According to Blasius, elsewhere, $\delta^* = 1.73 \sqrt{vx/U_\infty}$ (U_∞ free stream velocity, x distance from the leading edge of the plate). With the improved outer pressure distribution calculated as suggested above, there results as the superimposed pressure for an improved boundary calculation

$$p(x) = \frac{\rho U_\infty^2}{2} \left[1 + 1.35 \text{Re}(x) \right]^{-1}$$

($\text{Re}(x) = U_\infty x/\nu$). One does obtain here a considerable pressure correction since the new outer flow has a stagnation point, but one recognizes that the pressure correction has already dropped at $\text{Re}(x) = 75$ to 1 percent of the stagnation pressure. However, in the proximity of the leading edge of the plate (small $\text{Re}(x)$ - values) Blasius' boundary layer equation cannot be used. An improved calculation of the flow which in this range does not use the boundary layer approximations is still lacking. Thus for the time being the method of improvement for larger $\text{Re}(x)$ suggested in this report cannot be utilized for this example.

be $\bar{U} = c_0 t^n$ ($c_0 = \text{const.}$, $n \geq 0$, $t \geq 0$). Then the potential velocity $U_0(x, t)$ along the body boundaries can be represented in the form

$$U_0(x, t) = \begin{cases} 0 & \text{for } t \leq 0, \\ q(x)t^n & \text{for } t \geq 0 \end{cases} \quad (5.1)$$

We assume $\psi(x, y, t)$ to be the stream function of the boundary-layer flow defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ which is obtained when the pressure gradient to be calculated from equation (5.1) according to equation (1.2) is taken as a basis, on the strength of the boundary-layer equations (1.1), thus in the customary first approximation for very large Re-values. Generalizing the series developments set up by Blasius¹⁷ for the special cases $n = 0$ (sudden transition from state of rest to motion at constant velocity) and $n = 1$ (uniform acceleration) one obtains the result

$$\psi(x, y, t) = 2\sqrt{vt} \left[qt^n \zeta_{n,0}(\eta) + qq't^{2n+1} \zeta_{n,1}(\eta) + \dots \right], \quad (5.2)$$

thus a series development in powers of t^{n+1} (that is, in powers of the distance covered by the body). The appearing coefficient functions $\zeta_{n,0}(\eta)$, $\zeta_{n,1}(\eta)$, ... with $\eta = y/2\sqrt{vt}$ are universal functions of η . More details may be found in the appendix to this report.

From equation (5.2) follows

$$\left. \begin{aligned} u &= qt^n \zeta'_{n,0} + qq't^{2n+1} \zeta'_{n,1} + \dots, \\ v &= -2\sqrt{vt} \left[q't^n \zeta_{n,0} + (q'^2 + qq'')t^{2n+1} \zeta_{n,1} + \dots \right]. \end{aligned} \right\} \quad (5.3)$$

¹⁷Compare footnote 4 on page 3.

The boundary conditions are satisfied by virtue of

$$\left. \begin{aligned} \zeta_{n,0}(0) &= \zeta_{n,1}(0) = \dots = 0, \\ \zeta'_{n,0}(0) &= \zeta'_{n,1}(0) = \dots = 0, \\ \zeta'_{n,0}(\infty) &= 1, \quad \zeta_{n,1}(\infty) = \dots = 0, \end{aligned} \right\} \quad (5.4)$$

where there is always, here and later on

$$\zeta'_{n,\kappa} = \frac{d}{d\eta} \zeta_{n,\kappa}.$$

For the streamline displacement thickness δ^* there results

$$\delta^*(x,t) = 2 \sqrt{vt} \left[\alpha_n - q'(x) t^{n+1} \zeta_{n,1}(\infty) - \dots \right] \quad (5.5)$$

with

$$\alpha_n = \lim_{\eta \rightarrow \infty} \left[\eta - \zeta_{n,0}(\eta) \right]. \quad (5.6)$$

(A few numerical values α_n are given in the appendix.)

We consider as an example the flow about a circular cylinder. In the mode of notation of section 3, the velocity potential of the frictionless flow about the circular cylinder is given by

$$\phi_0 = \bar{U}(t) \left(r + \frac{R^2}{r} \right) \cos \vartheta \quad (5.7)$$

Hence the velocity at the periphery of the cylinder is calculated as

$$U_0(x, t) = q(x)t^n = 2\bar{U}(t) \sin \vartheta, \quad (5.8)$$

and the resulting pressure distribution at the body surface is

$$p_0(x, t) = p_d - \rho \left[2R(\cos \vartheta + 1) \frac{d\bar{U}}{dt} + 2\bar{U}^2(t) \sin^2 \vartheta \right], \quad (5.9)$$

where $p_d(t)$ is the pressure at the forward stagnation point. The potential-theoretical pressure drag of the length L of the cylinder at rest is therefore

$$W_{D0} = -L \int_0^{2\pi} p_0 \cos \vartheta R d\vartheta = 2\rho R^2 L \frac{d\bar{U}}{dt}. \quad (5.10)$$

Thus one obtains, as is known, an increase in inertia by double the amount of the inert mass of the fluid displaced by the cylinder itself. For the improved calculation of the outer potential flow there results

$$\delta^*(x, t) = 2 \sqrt{vt} \left[\alpha_n + 2 \frac{C_0}{R} \zeta_{n,1}(\infty) t^{n+1} \cos \vartheta + \dots \right]. \quad (5.11)$$

Following we limit ourselves to small times t for which the two first terms of the series development represent a sufficient approximation. In this approximation the line $y = \delta^*(x, t)$ practically represents a circle; that is to say, a circle with increasing radius r , the center of which, for $t > 0$, does not coincide with that of the cylinder cross-section, but travels slowly downstream. The equation of such a circle reads formally

$$r = b \cos \vartheta + a \left(1 - \frac{b^2}{a^2} \sin^2 \vartheta \right)^{\frac{1}{2}}$$

(a = radius, b = displacement of the center), and if $b \ll a$

$$r = a + b \cos \vartheta \left[+ O\left(\frac{b^2}{a}\right) \right]$$

We identify

$$\left. \begin{aligned} a(t) &= R + 2\alpha_n \sqrt{vt}, \\ b(t) &= \frac{4\sqrt{vt}}{R} \zeta_{n,1}^{(\infty)} C_0 t^{n+1}. \end{aligned} \right\} \quad (5.12)$$

If, temporarily, $\bar{r}, \bar{\vartheta}$, denote polar coordinates about the center of this circle in motion relative to the cylinder, the potential flow appertaining to the latter, the streamline of which is the line $r = R + \delta^*$, that is, $r = a(t)$, is given by the potential

$$\bar{\phi}_1 = \left(\bar{U} - \frac{db}{dt} \right) \left(1 + \frac{a^2}{\bar{r}^2} \right) \bar{r} \cos \bar{\vartheta}$$

thus the potential flow here required in the polar coordinate system r, ϑ fixed relative to the body by

$$\phi_1 = \bar{U} \left(1 + \frac{a^2}{r^2 - 2br \cos \vartheta + b^2} \right) (r \cos \vartheta - b) \quad (5.13)$$

At first we introduce dimensionless quantities as follows: We choose R as the characteristic length, and the time T the body requires to cover the distance R , starting from the beginning of motion $t = 0$, as the characteristic time, thus

$$T = \left[(n+1) \frac{R}{C_0} \right]^{\frac{1}{n+1}} \quad (5.14)$$

Hence results the characteristic velocity

$$\frac{R}{T} = \left(\frac{C_0 R^n}{n+1} \right)^{\frac{1}{n+1}}$$

and the Reynolds number Re formed with this velocity and the length R becomes¹⁸

$$Re = \left(\frac{R}{T} \right) \frac{R}{v} = \frac{R^2}{vT}. \quad (5.15)$$

By making dimensionless the variable lengths x, y, r, a, b by dividing by R , the time t by dividing by T , the velocity $\bar{U}(t)$ by dividing by R/T , the potential by dividing by R^2/T , and the pressure by dividing by $\rho R^2/T^2$ (we denote the dimensionless quantities by adding a wavy line), we obtain

$$\left. \begin{aligned} \tilde{a} &= 1 + 2\alpha_n Re^{-\frac{1}{2}\tilde{t}^{\frac{1}{2}}}, \\ \tilde{b} &= 4(n+1)\zeta_{n,1}(\infty) Re^{-\frac{1}{2}\tilde{t}^{\frac{1}{2}}}; \end{aligned} \right\} \quad (5.12a)$$

$$\tilde{\phi}_1 = - (n+1)\tilde{t}^n \left(1 + \frac{\tilde{a}^2}{\tilde{r}^2 + 2\tilde{b}\tilde{r}\cos\tilde{x} + \tilde{b}^2} \right) (\tilde{r}\cos\tilde{x} + \tilde{b}). \quad (5.13a)$$

¹⁸If one forms the Reynolds number Re_1 with the length \sqrt{vT} one obtains

$$Re_1 = (R/T) \sqrt{vT}/v = Re^{\frac{1}{2}}$$

Furthermore,

$$\eta = \frac{y}{2\sqrt{vt}} = \frac{\tilde{y}}{2} \text{Re}^{\frac{1}{2}\tilde{t}} \tilde{t}^{-\frac{1}{2}}$$

and

$$\frac{\delta^*}{R} = \frac{\tilde{t}^{\frac{1}{2}}}{\text{Re}^{\frac{1}{2}}} \left[2\alpha_n - 4(n+1)\zeta_{n,1}(\infty)\tilde{t}^{n+1} \cos \tilde{x} \right]. \quad (5.11a)$$

We consider first the case $n = 1$ of uniform acceleration from a state of rest. Following, P_1 without more precise data will represent those additive portions of the pressure $\tilde{p}_1 = p_1/(\rho R^2/T^2)$ which do not make any contributions to the pressure and are therefore not of interest to us. Furthermore, terms of the order $O(\delta^2/R^2)$ compared with 1 are neglected. One obtains for the pressure calculation according to the Bernoulli equation at the distance from the wall $\tilde{y} = \delta/R$

$$\begin{aligned} -\frac{\partial \tilde{\phi}_1}{\partial \tilde{t}} &= 4 \left(1 + 3\alpha_1 \text{Re}^{\frac{1}{2}\tilde{t}} \tilde{t}^{\frac{1}{2}} \right) \cos \tilde{x} + P_1 + O\left(\frac{\delta^2}{R^2}\right) \\ -\frac{1}{2} \left(\frac{\partial \tilde{\phi}_1}{\partial \tilde{r}} \right)^2 &= O\left(\frac{\delta^2}{R^2}\right) \\ -\frac{1}{2} \left(\frac{1}{\tilde{r}} \frac{\partial \tilde{\phi}_1}{\partial \tilde{x}} \right)^2 &= 16 \frac{\delta}{R} \tilde{t}^2 \sin^2 \tilde{x} + P_2 + O\left(\frac{\delta^2}{R^2}\right). \end{aligned}$$

Thus one obtains

$$\tilde{p}_1\left(\tilde{x}, \frac{\delta}{R}, \tilde{t}\right) = 4 \left(1 + 3\alpha_1 \text{Re}^{\frac{1}{2}\tilde{t}} \tilde{t}^{\frac{1}{2}} \right) \cos \tilde{x} + 16 \frac{\delta}{R} \tilde{t}^2 \sin^2 \tilde{x} + P_3 + O\left(\frac{\delta^2}{R^2}\right). \quad (5.16)$$

According to equation (4.4)

$$\tilde{p}_1(\tilde{x}, 0, \tilde{\tau}) = \tilde{p}_1\left(\tilde{x}, \frac{\delta}{R}, \tilde{\tau}\right) - \int_0^{\delta/R} \tilde{u}^2 d\tilde{y}. \quad (4.4a)$$

In the small times of interest to us there is

$$\tilde{u} = 4 \tilde{\tau} \sin \tilde{x} \zeta'_{1,0}(\eta) - 16 \tilde{\tau}^3 \sin \tilde{x} \cos \tilde{x} \zeta'_{1,1}(\eta),$$

therefore

$$\left. \begin{aligned} - \int_0^{\delta/R} \tilde{u}^2 d\tilde{y} = & - 16 \tilde{\tau}^3 \sin^2 \tilde{x} \int_0^{\delta/R} \zeta'^2_{1,0} d\tilde{y} + \\ & 128 \tilde{\tau}^4 \sin^2 \tilde{x} \cos \tilde{x} \int_0^{\delta/R} \zeta'_{1,0} \zeta'_{1,1} d\tilde{y} - \\ & 256 \tilde{\tau}^6 \sin^2 \tilde{x} \cos^2 \tilde{x} \int_0^{\delta/R} \zeta'^2_{1,1} d\tilde{y}. \end{aligned} \right\} \quad (5.17)$$

Because of $\lim_{\eta \rightarrow \infty} \zeta'_{1,0}(\eta) = 1$, $\lim_{\eta \rightarrow \infty} \zeta'_{1,1}(\eta) = 0$ the two last integrals

on the right are, if δ/R is chosen so large that these asymptotic values are attained with the desirable accuracy, numbers independent of the variation of the boundary-layer edge $\delta(\tilde{x}, \tilde{\tau})$ with \tilde{x} . The first integral at the right, together with the corresponding second term at the right in equation (5.16), may also be combined into an expression independent of $\delta(\tilde{x}, \tilde{\tau})$, namely the expression

$$16 \tilde{\tau}^2 \sin^2 \tilde{x} \int_0^{\infty} (1 - \zeta'^2_{1,0}) d\tilde{y}.$$

Thus the variation of $\delta(\tilde{x}, \tilde{t})$ over \tilde{x} does not play any role in judging what terms make contributions to the pressure drag which is as it should be. If one finally inserts

$$d\tilde{y} = 2 \operatorname{Re}^{-\frac{1}{2}} \tilde{t}^{\frac{1}{2}} d\eta,$$

all together one obtains, therefore,

$$\left. \begin{aligned} \tilde{p}(\tilde{x}, 0, \tilde{t}) = & 4 \left(1 + 3\alpha_1 \operatorname{Re}^{-\frac{1}{2}} \tilde{t}^{\frac{1}{2}} \right) \cos \tilde{x} \\ & + 256 \sin^2 \tilde{x} \cos \tilde{x} \operatorname{Re}^{-\frac{1}{2}} \tilde{t}^{\frac{9}{2}} \int_0^\infty \zeta'_{1,0} \zeta'_{1,1} d\eta + P_4 + O(\operatorname{Re}^{-1}). \end{aligned} \right\} \quad (5.18)$$

Thus the following pressure drag of a circular cylinder of the length L for small times after start of the motion and in first order for large Re -values results

$$\frac{W_D}{2\rho\pi R^2 L C_0} = 1 + \frac{1}{\tilde{t}^2} \operatorname{Re}^{-\frac{1}{2}} (3\alpha_1 + 16 \tilde{t}^4 \int_0^\infty \zeta'_{1,0} \zeta'_{1,1} d\eta). \quad (5.19a)$$

$2\rho\pi R^2 L C_0 = W_{D_0}$ is the potential-theoretical pressure drag. (Compare equation (5.10).) According to Blasius' calculations $3\alpha_1 = 2/\sqrt{\pi} (=1.128)$, furthermore, according to the author's numerical evaluation

$$\int_0^\infty \zeta'_{1,0} \zeta'_{1,1} d\eta = 0.09804$$

Thus the result in dimensional form is

$$W_D = W_{D0} \left[1 + \frac{\sqrt{\nu t}}{R} \left(\frac{2}{\sqrt{\pi}} + 0.392 \frac{C_0^2}{R^2} t^4 \right) \right]. \quad (5.19)$$

As friction drag

$$W_R = \rho \nu \int_0^{2\pi} \frac{\partial u(x, 0, t)}{\partial y} \sin \frac{x}{R} dx$$

there results from Blasius' calculation results¹⁹

$$W_R = W_{D0} \frac{\sqrt{\nu t}}{R} \left(\frac{2}{\sqrt{\pi}} - 0.029 \frac{C_0^2}{R^2} t^4 \right). \quad (5.20)$$

For sufficiently small times $t(\tilde{t} \ll 1)$ thus $W_R = W_D - W_{D0}$, that is, the friction drag increases immediately after start of the motion according to the same law as the contribution of the skin friction to the pressure drag.

The total drag becomes

$$W = W_D + W_R = W_{D0} \left[1 + \frac{\sqrt{\nu t}}{R} \left(\frac{4}{\sqrt{\pi}} + 0.363 \frac{C_0^2}{R^2} t^4 \right) \right]. \quad (5.21)$$

In figure 2 we represented these results using the dimensionless $\tilde{t} = t/T$ (with $T = \sqrt{2R/C_0}$ for the present case $n = 1$), thus the relations

$$\frac{W_D}{W_{D0}} = 1 + \frac{\sqrt{\nu T}}{R} \tilde{t}^{\frac{1}{2}} (1.128 + 1.569 \tilde{t}^4), \quad (5.19_1)$$

¹⁹In order to obtain the term with t^4 , one must here include in equation (5.2) also the third term of the series. The necessary data may be found in Blasius' report, footnote 4.

$$\frac{W_R}{W_{D0}} = \frac{\sqrt{vT}}{R} \tilde{t}^{\frac{1}{2}} (1.128 - 0.116\tilde{t}^4), \quad (5.20_1)$$

$$\frac{W}{W_{D0}} = 1 + \frac{\sqrt{vT}}{R} \tilde{t}^{\frac{1}{2}} (2.257 + 1.453\tilde{t}^4). \quad (5.21_1)$$

Our formulas can be applied with good approximation only for very small times after start of the motion (solidly drawn parts of the curves), because we had, for the sake of simplicity, taken into consideration only a few terms of the development (5.2); but of course, with a little calculation expenditure they can be easily improved, at least so far that they are valid up to times shortly after setting-in of the separation.

The value $\tilde{t} = \tilde{t}_A (= \sqrt{0.586}) = 0.766$ plotted in figure 2 indicates the time of the start of separation, in first approximation, at the rearward stagnation point according to Blasius. (Compare also appendix.)

For an arbitrary integral $n > 0$ one obtains correspondingly as pressure drag with $\overline{\Pi}(t) = C_0 t^n$ and $W_{D0} = 2\rho\pi R^2 L d\overline{U}/dt$

$$\frac{W_D}{W_{D0}} = 1 + \frac{\sqrt{vt}}{R} \left[\left(2 + \frac{1}{n}\right) \alpha_n + \frac{4C_0^2}{nR^2} t^{2n+2} \int_0^\infty \xi'_{n,0} \xi'_{n,1} d\eta \right] \quad (5.22)$$

For the calculation of the coefficient of t^{2n+2} sufficient numerical data concerning $\xi'_{n,1}(\eta)$ are lacking so far. For this reason we limit ourselves in our numerical statements to the first term of the develop-

ment with time (term with $t^{\frac{1}{2}}$) and put the question whether the law $W_R = W_D - W_{D0}$ obtained above for $n = 1$, for times immediately after start of the motion, is valid also for arbitrary $n > 0$. The friction drag has for these small times the value

$$W_R = W_{D0} \frac{\sqrt{vt}}{R} \frac{\xi''_{n,0}(0)}{2n}. \quad (5.23)$$

In the appendix the exact expressions for $\xi_{n,0}(\eta)$ are derived.

However, the relation

$$2(2n + 1)\alpha_n = \xi_{n,0}''(0) = 2^{2n} \frac{(n!)^2}{(2n)!} \frac{2}{\sqrt{\pi}} \quad (5.24)$$

is valid; we produce a very simple proof for it in the appendix (one may confirm it also with the aid of the tables of the appendix we calculated for $n = 1, 2, 3$, and 4). Thus, for very small times after start of the motion, there applies indeed

$$\left. \begin{aligned} w_R &= w_D - w_{D0} \\ \text{viz:} \quad &= w_{D0} \frac{\sqrt{vt}}{R} 2^{2n} \frac{n!(n-1)!}{(2n)! \sqrt{\pi}} \quad (n > 0). \end{aligned} \right\} \quad (5.25)$$

(One has $2^{2n} n!(n-1)!/(2n)! \sqrt{\pi} = 1.1284$ for $n = 1$; 0.7522 for $n = 2$; 0.6018 for $n = 3$; 0.5158 for $n = 4$, etc.)

One may now consider more general laws of motion of the form

$$\bar{U}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ f(t) & \text{for } t \geq 0 (f(0) = 0) \end{cases}$$

and carry out corresponding calculations. If one presupposes that the function $f(t)$ defined in $t \geq 0$ can be developed into a Taylor series around $t = 0$ which converges for the small times $0 \leq t \ll t_A$ after start of the motion which are of interest to us, one may attain the result quite analogously with a series expression correspondingly generalized compared to equation (5.2). One can interpret the law $\bar{U} = C_0 t^n$ which has been valid so far as the first term of the development with time of such a general law of motion. Hence it follows

that for all these laws of motion the portion of the pressure drag caused by friction $W_D - W_{D0}$ increases immediately after the start of motion according to the same law (5.25) as the friction drag. This is a noteworthy quality of the circular cylinder.

One may also include the case of a sudden start of motion in these considerations; it is true that one must then accept, corresponding to this degeneration of the form of motion at the time $t = 0$, infinite pressures and drags at the time $t = 0$. With $\bar{U} \equiv C_0 = \text{const.}$ for $t \geq 0$ one obtains

$$W_D = 2\rho\pi R L C_0 \sqrt{\frac{\nu}{t}} \left[\alpha_0 + \frac{4C_0^2}{R^2} t^2 \int_0^\infty \zeta'_{0,0} \zeta'_{0,1} d\eta \right] \quad (5.26)$$

and

$$W_R = 2\rho\pi R L C_0 \sqrt{\frac{\nu}{t}} \left[\frac{\zeta''_{0,0}(0)}{2} + \frac{\zeta''_{0,2a}(0)}{2} \frac{\bar{U}_0^2}{R^2} t^2 \right] \quad (5.27)$$

The function $\zeta_{0,2a}(\eta)$ is explained at the end of the appendix. We did not numerically determine the coefficients at t^2 in equations (5.26) and (5.27). Because of $\alpha_0 = \frac{1}{2} \zeta'_{0,0}(0) (= 1/\sqrt{\pi} = 0.5642)$ the law stated above $W_R = W_D - W_{D0}$ is valid also in this limiting case $n = 0$ of the sudden start of motion, for times immediately after the start of motion. Here in particular $W_{D0} = 0$ (d'Alembert's paradox).

APPENDIX

A FEW CALCULATIONS REGARDING THE DEVELOPMENT OF THE BOUNDARY LAYERS

A few calculations will be given in this appendix which yield, among other data, those required for the preceding investigation (section 5) concerning the basic functions $\zeta_{n,i}(\eta)$ of the unsteady boundary layers; for the rest, they represent merely an extension of the related calculations by Blasius. We give these calculations apart from the previous considerations, first, because they would have disrupted the connection there, and second, because the data and tables attained are of interest in their own right.

As assumed in section 5, let a velocity proportional to $t^n (n \geq 0)$ be imparted to a body from a state of rest relative to the surrounding indefinitely extended fluid, beginning at $t = 0$. The potential-theoretical circumferential velocity $U(x, 0, t) = U_0(x, t)$ then has the form (5.1). For calculation of the boundary layer development from $t = 0$, if a generalization is made of the series set up by Blasius for $n = 0$ and $n = 1$, the expression

$$\psi(x, \eta, t) = 2\sqrt{\nu t} \sum_{\lambda=0}^{\infty} t^{n+\lambda(n+1)} \chi_{n,\lambda}(x, \eta) \quad (1)$$

with

$$\eta = y/2\sqrt{\nu t}$$

for the stream function ψ of the boundary-layer flow is obtained and one obtains for the functions $\chi_{n,\lambda}$ by substituting equation (1) into the boundary layer differential equation (1.1) a system of differential equations solved by recursion; we limit ourselves here to the two first equations of this system which read

$$\left. \begin{aligned} \frac{\partial^3 \chi_{n,0}}{\partial \eta^3} + 2\eta \frac{\partial^2 \chi_{n,0}}{\partial \eta^2} - 4n \frac{\partial \chi_{n,0}}{\partial \eta} &= -4nq(x), \\ \frac{\partial^3 \chi_{n,1}}{\partial \eta^3} + 2\eta \frac{\partial^2 \chi_{n,1}}{\partial \eta^2} - 4(2n+1) \frac{\partial \chi_{n,1}}{\partial \eta} &= 4 \left(\frac{\partial \chi_{n,0}}{\partial \eta} \frac{\partial^2 \chi_{n,0}}{\partial x \partial \eta} - \frac{\partial \chi_{n,0}}{\partial x} \frac{\partial^2 \chi_{n,0}}{\partial \eta^2} - qq' \right). \end{aligned} \right\} \quad (2)$$

We assume first n (later $2n$) to be an integer. Then the solutions may be represented with the aid of Hermite polynomials. With the statements

$$\left. \begin{aligned} \chi_{n,0} &= q(x)\zeta_{n,0}(\eta), \\ \chi_{n,1} &= q(x)q'(x)\zeta_{n,1}(\eta) \end{aligned} \right\} \quad (3)$$

one obtains from equation (2) the ordinary differential equations

$$\zeta_{n,0}'''' + 2\eta\zeta_{n,0}''' - 4n\zeta_{n,0}' = -4n, \quad (4)$$

$$\zeta_{n,1}'''' + 2\eta\zeta_{n,1}''' - 4(2n+1)\zeta_{n,1}' = 4(\zeta_{n,0}'^2 - \zeta_{n,0}\zeta_{n,0}'' - 1). \quad (5)$$

The boundary conditions to be satisfied by $\zeta_{n,0}$ and $\zeta_{n,1}$ are formulated in equation (5.4)

In the case of a plane wall moving in its own plane, $\psi = 2\sqrt{vt} \, qt^n \zeta_{n,0}(\eta)$ represents the complete solution (because of $q = \text{const.}$), not only in the boundary-layer theory approximation set up here, but in strict fulfillment of the complete Navier-Stokes equations.²⁰ The calculation of $\zeta_{n,0}$ may take place as follows. The temporary splitting-off

$$\zeta'_{n,0}(\eta) = 1 - e^{-\eta^2} \phi_n(\eta) \quad (6)$$

transforms equation (4) into

$$\phi_n'' - 2\eta\phi_n' - 2(2n+1)\phi_n = 0 \quad (7)$$

²⁰Because of $U_0 = U_0(t)$, $u = u(y,t)$ and $v \equiv 0$, the Navier-Stokes equations are for these motions simplified to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\partial U_0}{\partial t}$$

The analogy between $U_0 - u$ and the corresponding solutions of the problem of heat conduction has been known for a long time; it offered one of the few possibilities of attaining exact solutions of the Navier-Stokes equations. For the rest, one can see for the present problem that the first term $qt\zeta'_{n,0}$ of the series development following from equation (1) for u as a solution of the above equation approximately satisfies the boundary-layer equation in the sense that only the terms of highest order are taken into consideration for small times after start of the motion, whereas the quadratic inertia terms are neglected. The iterative improvement of this first approximation for small times then yields step by step the ascending terms of the series we set up formally at the outset. This consideration led Blasius, (elsewhere), at the time to his special series formulations for $n = 0$ and $n = 1$.

For every integral $2n \geq 0$ the general solution of this differential equation²¹ is

$$\varphi_n = \left(\frac{d}{d\eta}\right)^{2n} \left[e^{\eta^2} \left(C_1 + C_2 \int_{\infty}^{\eta} e^{-\eta^2} d\eta \right) \right] = \left(\frac{d}{d\eta}\right)^{2n} \varphi_0 \quad (8)$$

(C_1, C_2 are integration constants). As is well known, the Hermite polynomials $H_m(x)$ are given by

$$H_m(x) = e^{x^2} \left(-\frac{d}{dx}\right)^m e^{-x^2} = \sum_{0 \leq \kappa \leq \frac{m}{2}} (-1)^{\kappa} \binom{m}{2\kappa} \frac{(2\kappa)!}{\kappa!} (2x)^{m-2\kappa} \quad (9)$$

(in the form originally given by Hermite). For further use we also put $H_m(ix) = i^m \tilde{H}_m(x)$, thus

$$\tilde{H}_m(x) = e^{-x^2} \left(\frac{d}{dx}\right)^m e^{x^2} = \sum_{0 \leq \kappa \leq \frac{m}{2}} \binom{m}{2\kappa} \frac{(2\kappa)!}{\kappa!} (2x)^{m-2\kappa}, \quad (10)$$

furthermore

$$\phi_0(\eta) = \frac{2}{\sqrt{\pi}} \int_{\infty}^x e^{-x^2} dx, \quad \phi_{\kappa}(x) = \frac{d^{\kappa} \phi_0}{dx^{\kappa}} \quad (11)$$

²¹Compare, for instance, E. Kamke "Differentialgleichungen: Lösungsmethoden und Lösungen, I Gewöhnliche Differentialgleichungen" (Differential equations: Methods of solutions and solutions, I. ordinary differential equations) Leipzig 1942, Part C, No. 2, 41, and put there $x = i\sqrt{2}\eta$.

(so that in particular $1 + \phi_0 = \phi$ represents the error integral, ϕ_1 the error function). By using these expressions, a simple recalculation from equation (8) gives the general solution of equation (4) as

$$\zeta'_{n,0}(\eta) = 1 - \tilde{H}_{2n}(\eta) \left[C_1 + \frac{C_2^*}{2^0} \phi(\eta) \right] + \frac{C_2^* \phi_1(\eta)}{2^1} \sum_{\kappa=1}^{2n} \binom{2n}{\kappa} (-1)^\kappa \tilde{H}_{2n-\kappa}(\eta) H_{\kappa-1}(\eta) \quad (12)$$

with $C_2^* = \frac{\sqrt{\pi}}{2} C_2$.

$C_1 = 0$ because of $\zeta'_{n,0}(\infty) = 1$ and $C_2^* = -n!/(2n)!$ because of $\zeta'_{n,0}(0) = 0$ and $\tilde{H}_{2n}(0) = (2n)!/n!$. [The polynomial sum in equation (12) disappears for $\eta = 0$ and integral n term by term, since either $2n - \kappa$ or $\kappa - 1$ is an odd number]. Thus the desired particular integral of (4)²² reads

$$\zeta'_{n,0}(\eta) = 1 + \frac{n!}{(2n)!} \left[\tilde{H}_{2n}(\eta) \phi_0(\eta) - \phi_1(\eta) \sum_{\kappa=1}^{2n} \binom{2n}{\kappa} (-1)^\kappa \tilde{H}_{2n-\kappa}(\eta) H_{\kappa-1}(\eta) \right]. \quad (13)$$

Because of $\phi_\kappa(\eta) = (-1)^{\kappa-1} \phi_1(\eta) H_{\kappa-1}(\eta)$ the solution may also be written in the following form which is more elegant than equation (13) but less serviceable for practical calculation

$$\zeta'_{n,0}(\eta) = 1 + \frac{n!}{(2n)!} \sum_{\kappa=0}^{2n} \binom{2n}{\kappa} \phi_\kappa(\eta) \tilde{H}_{2n-\kappa}(\eta). \quad (13a)$$

²²The direct and elementary derivation of the solution for integral $n \geq 0$, and therewith for integral $2n \geq 0$, compare below, fails to work if n does not have this property. But in that case, too, the solutions are easily found if one makes use of the analogy to the corresponding solutions of linear heat conduction (compare footnote 20 to this report) and represents the solution according to the singularity method.

According to equation (13) one has in particular

$$\left. \begin{aligned} \zeta'_{0,0}(\eta) &= 1 + \phi_0(\eta) = \phi(\eta), \\ \zeta'_{1,0}(\eta) &= 1 + (2\eta^2 + 1)\phi_0(\eta) + \eta\phi_1(\eta), \\ \zeta'_{2,0}(\eta) &= 1 + \frac{1}{3} \left[(4\eta^4 + 12\eta^2 + 3)\phi_0(\eta) + (2\eta^2 + 5)\eta\phi_1(\eta) \right], \\ \zeta'_{3,0}(\eta) &= 1 + \frac{1}{15} \left[8\eta^6 + 60\eta^4 + 90\eta^2 + 15\phi_0(\eta) + \right. \\ &\quad \left. (4\eta^4 + 28\eta^2 + 33)\eta\phi_1(\eta) \right], \\ \zeta'_{4,0}(\eta) &= 1 + \frac{1}{105} \left[16\eta^8 + 224\eta^6 + 840\eta^4 + 840\eta^2 + 105\phi_0(\eta) + \right. \\ &\quad \left. (8\eta^6 + 108\eta^4 + 370\eta^2 + 279)\eta\phi_1(\eta) \right]. \end{aligned} \right\} \quad (13b)$$

Thence one obtains by elementary quadratures $\zeta_{n,0}(\eta)$, likewise expressed by ϕ_0 and ϕ_1 with polynomial coefficients. The numerical evaluation is reproduced in table 1²³. Because of the special importance of the solutions $\zeta'_{n,0}(\eta)$ as boundary-layer profiles u/qt^n in the case of the plane wall (compare above) we represented them in figure 3. [Owing to $\delta^* = 2\sqrt{vt} \lim_{\eta \rightarrow \infty} (\eta - \zeta_{n,0}(\eta)) = 2\alpha_n \sqrt{vt}$ the stream-line displacement thickness δ^* can easily be taken from the numerical cal table 1; compare also table 2.]

Since it follows directly from equation (7), by single differentiation, that the general integral $\phi_{n+\frac{1}{2}}$ is the first derivative of the general integral ϕ_n with respect to η , it is, with the boundary conditions satisfied, easy to find as expressions for the basic functions $\zeta_{n+\frac{1}{2},0}(\eta)$ with the integral n

$$\left. \zeta'_{n+\frac{1}{2},0} = 1 - \frac{1}{\zeta''_{n,0}(0)} \left[\zeta''_{n,0} + 2\eta(1 - \zeta'_{n,0}) \right] \right\} \quad (14)$$

²³All numerical calculations were performed by Miss Ursula Ludewig.

Therein

$$\zeta''_{n,0}(0) = 2^{2n} \frac{(n!)^2}{(2n)!} \frac{2}{\sqrt{\pi}}. \quad (15)$$

In order to calculate the fundamental functions of the first order $\zeta_{n,1}(\eta)$, the total course of $\zeta_{n,0}(\eta)$ must be known (according to equation (5)). Only the additional knowledge of $\zeta''_{n,1}(0)$ is required for the problem which is of foremost interest, the question regarding location and time of a possible separation of the boundary layer in first approximation.

For because of

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{t^n}{2\sqrt{vt}} \left[q \zeta''_{n,0}(0) + q q' t^{n+1} \zeta''_{n,1}(0) + \dots \right]$$

one obtains in first approximation the connection

$$t = \left[\frac{-\zeta''_{n,0}(0)}{q'(x) \zeta''_{n,1}(0)} \right]^{\frac{1}{n+1}} \quad (16)$$

for location x and time t of the separation. On the other hand, we are interested in $\zeta_{n,1}(\infty)$ (compare section 5), with a view to the calculation of the displacement thickness. These two data can be determined without solution of the differential equation (5) by a well-known method as follows: $\vartheta(\eta)$ is assumed to be a function of η in $0 \leq \eta \leq \infty$ provided with the continuity properties required for the following calculation. By partial integration one obtains the following relations. If L_n and M_n are the differential operators

$$L_n = \frac{d^2}{d\eta^2} + 2\eta \frac{d}{d\eta} - 4(2n+1),$$

$$M_n = \frac{d^2}{d\eta^2} - 2\eta \frac{d}{d\eta} - 4\left(2n + \frac{3}{2}\right),$$

one has

$$\int_0^\infty \vartheta(\eta) L_n[\xi'_{n,1}] d\eta = \left[\vartheta \xi''_{n,1} - \vartheta' \xi'_{n,1} + 2\eta \vartheta \xi'_{n,1} \right]_0^\infty + \int_0^\infty \xi'_{n,1} M_n[\vartheta] d\eta, \quad (17)$$

$$\int_0^\infty \vartheta(\eta) L_n[\xi'_{n,1}] d\eta = \left[\vartheta \xi''_{n,1} - \vartheta' \xi'_{n,1} + 2\eta \vartheta \xi'_{n,1} + \xi_{n,1} M_n[\vartheta] \right]_0^\infty - \int_0^\infty \xi_{n,1} \frac{d}{d\eta} M_n[\vartheta] d\eta. \quad (18)$$

We choose, therefore, $\vartheta = \vartheta_{n,0}(\eta)$ so that

$$M_n[\vartheta_{n,0}] = 0 \quad \text{with} \quad (17a)$$

$$\vartheta_{n,0}(0) = -1, \quad \vartheta_{n,0}(\infty) = 0$$

If then equation (5) and the boundary conditions valid for $\xi_{n,1}(\eta)$ are taken into consideration the result is

$$\xi''_{n,1}(0) = \int_0^\infty \vartheta_{n,0}(\eta) \left(\xi_{n,0}'^2 - \xi_{n,0} \xi_{n,0}'' - 1 \right) d\eta. \quad (17b)$$

On the other hand we choose $\vartheta = \vartheta_{n,\infty}(\eta)$ so that

$$M_n[\vartheta_{n,\infty}] = 1 \quad \text{with} \quad (18a)$$

$$\vartheta_{n,\infty}(0) = 0, \quad \vartheta_{n,\infty}(\infty) \text{ finite}$$

Then one obtains, in analogy

$$\zeta_{n,1}^{(\infty)} = \int_0^\infty \vartheta_{n,\infty}(\eta) \left(\zeta_{n,0}'^2 - \zeta_{n,0}'' \zeta_{n,0} - 1 \right) d\eta. \quad (18b)$$

It is easily confirmed that

$$\vartheta_{n,\infty}(\eta) = \frac{-1}{2(4n+3)} \left[\vartheta_{n,0}(\eta) + 1 \right], \quad (19)$$

furthermore by comparison with equation (7)

$$\vartheta_{n,0}(\eta) = e^{\eta^2} \left[\zeta_{2n+1,0}'(\eta) - 1 \right], \quad (20)$$

Therewith the desired functions $\vartheta_{n,0}$ and $\vartheta_{n,\infty}$ are traced back to the known basic functions of zero order $\zeta_{2n+1,0}$.

Numerical evaluation yielded for $n = 0, 1, 2, 3$, and 4 all together the data here of interest given in table 2. It also shows the numerical values of α_n . For the n -values $1, 2, 3$, and 4 one finds the law (5.24) confirmed. A general proof of this law may be produced with a few calculations on the basis of the known expressions for $\zeta_{n,0}(\eta)$. A much simpler proof of the relation (5.24) will be presented below. As mentioned above, the expression

$$u = qt^n \zeta_{n,0}'(\eta)$$

is the strict solution for the boundary-layer profile on a plane wall with $U = qt^n$ ($q = \text{const.}$) outside of the boundary layer. The wall shearing stress according to the momentum theorem of the boundary layers is generally

$$\tau_0 = U \frac{\partial}{\partial x} \int_0^\delta \rho u dy - \frac{\partial}{\partial x} \int_0^\delta \rho u^2 dy - \int_0^\delta \rho \frac{\partial u}{\partial t} dy - \delta \frac{\partial p}{\partial x}, \quad (21)$$

thus for the above flow, due to the velocity distribution $u(\eta, t)$ being independent of x as well as due to $\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t}$ and with

$$\alpha_n = \lim_{\eta \rightarrow 0} [\eta - \zeta_{n,0}(\eta)]$$

$$\tau_0 = \rho \frac{\partial}{\partial t} \int_0^\delta (U - u) dy = \rho \frac{\partial}{\partial t} (U\delta^*) = \rho q(2n+1) \alpha_n \sqrt{v} t^{n-\frac{1}{2}}. \quad (22)$$

On the other hand

$$\tau_0 = \mu \frac{\partial u(0, t)}{\partial y} = \rho q \zeta_{n,0}''(0) \frac{\sqrt{v}}{2} t^{n-\frac{1}{2}}, \quad (23)$$

and thus in combination with equation (15), as asserted,

$$2(2n+1)\alpha_n = \zeta_{n,0}''(0) = \frac{2^{2n+1}(n!)^2}{(2n)! \sqrt{\pi}}. \quad (5.24)$$

In the case of the flow about a circular cylinder moved rectilinearly out of a state of rest, investigated in section 5 [compare equation (5.8)] one has $q(x) = 2C_0 \sin x/R$ and therefore $-q'(x)_{\max} = 2C_0/R$ for $x/R = \pi$. Thus the separation starts according to equation (16) at the rearward stagnation point at the time

$$t_A = \left[R \zeta_{n,0}''(0) / 2C_0 \zeta_{n,1}''(0) \right]^{\frac{1}{n+1}}$$

(compare table 3). The distance covered by the cylinder during that time is

$$S_A = C_0 t_A^{n+1} / (n+1) = R \zeta_{n,0}''(0) / 2(n+1) \zeta_{n,1}''(0)$$

Finally we want to give a few indications where to find further data regarding the basic functions of the plane nonsteady boundary-layer flow. The series development (5.2) written down up to the third term reads

$$\psi = 2 \sqrt{vt} \, qt^n \left[\xi_{n,0} + t^{n+1} q' \xi_{n,1} + t^{2(n+1)} (q'^2 \xi_{n,2a} + qq'' \xi_{n,2b}) + \dots \right].$$

Blasius gives, in addition to the functions $\xi_{0,0}(\eta)$ and $\xi_{1,0}(\eta)$ calculated above, the rigorous solutions $\xi_{0,1}(\eta)$ and $\xi_{1,1}(\eta)$.

Beyond that, he calculates the numerical values $\xi_{1,2a}''(0)$ and $\xi_{1,2b}''(0)$ which are of interest for the determination of the separation.

S. Goldstein and L. Rosenhead²⁴ give the exact expressions for $\xi_{0,2a}(\eta)$ and $\xi_{0,2b}(\eta)$. These integrals were, by the way, numerically determined before by Boltze²⁵ on the occasion of treatment of the corresponding problem $n = 0$ of rotationally symmetrical flows which seems to have escaped the attention of the authors.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

²⁴Compare footnote 6 of this report.

²⁵Compare footnote 5 of this report.

TABLE 1

η	$\zeta_{0,0}$	$\zeta'_{0,0}$	$\zeta''_{0,0}$	$\zeta_{1,0}$	$\zeta'_{1,0}$	$\zeta''_{1,0}$	$\zeta_{2,0}$	$\zeta'_{2,0}$	$\zeta''_{2,0}$	$\zeta_{3,0}$	$\zeta'_{3,0}$	$\zeta''_{3,0}$	$\zeta_{4,0}$	$\zeta'_{4,0}$	$\zeta''_{4,0}$
0	0	0	1.1284	0	0	2.2568	0	0	3.0090	0	0	3.6108	0	0	4.1266
0.1	0.0056	0.1125	1.1172	0.0106	0.2064	1.8793	0.0138	0.2638	2.2941	0.0162	0.3067	2.5762	0.0182	0.3415	2.7858
0.2	0.0224	0.2227	1.0841	0.0401	0.3773	1.5464	0.0506	0.4638	1.7298	0.0584	0.5245	1.8184	0.0646	0.5710	1.8608
0.3	0.0500	0.3287	1.0313	0.0851	0.5172	1.2569	0.1049	0.6139	1.2896	0.1189	0.6718	1.2695	0.1298	0.7235	1.2296
0.4	0.0879	0.4284	0.9615	0.1426	0.6301	1.0085	0.1721	0.7251	0.9501	0.1923	0.7835	0.8763	0.2075	0.8237	0.8035
0.5	0.1355	0.5205	0.8788	0.2103	0.7201	0.7986	0.2489	0.8066	0.6916	0.2745	0.8564	0.5978	0.2934	0.8889	0.5192
0.6	0.1917	0.6039	0.7872	0.2860	0.7910	0.6237	0.3327	0.8656	0.4972	0.3628	0.9059	0.4031	0.3845	0.9307	0.3316
0.7	0.2559	0.6778	0.6913	0.3680	0.8459	0.4804	0.4215	0.9077	0.3529	0.4552	0.9390	0.2685	0.4790	0.9573	0.2093
0.8	0.3270	0.7421	0.5950	0.4548	0.8880	0.3647	0.5138	0.9374	0.2473	0.5502	0.9610	0.1766	0.5757	0.9740	0.1305
0.9	0.4040	0.7969	0.5020	0.5452	0.9197	0.2728	0.6087	0.9581	0.1709	0.6471	0.9753	0.1147	0.6736	0.9844	0.0804
1.0	0.4861	0.8427	0.4151	0.6384	0.9432	0.2010	0.7052	0.9724	0.1166	0.7451	0.9846	0.0736	0.7724	0.9907	0.0489
1.1	0.5723	0.8802	0.3365	0.7337	0.9604	0.1459	0.8030	0.9820	0.0784	0.8439	0.9905	0.0466	0.8717	0.9946	0.0294
1.2	0.6619	0.9103	0.2673	0.8304	0.9728	0.1042	0.9015	0.9884	0.0520	0.9432	0.9942	0.0291	0.9713	0.9969	0.0175
1.3	0.7541	0.9340	0.2082	0.9281	0.9816	0.0733	1.0006	0.9927	0.0340	1.0427	0.9966	0.0179	1.0710	0.9982	0.0102
1.4	0.8485	0.9523	0.1589	1.0266	0.9878	0.0507	1.1000	0.9954	0.0219	1.1425	0.9980	0.0109	1.1709	0.9990	0.0059
1.5	0.9444	0.9661	0.1189	1.1256	0.9920	0.0345	1.1996	0.9972	0.0139	1.2423	0.9988	0.0065	1.2708	0.9994	0.0034
1.6	1.0416	0.9763	0.0872	1.2250	0.9948	0.0231	1.2994	0.9983	0.0087	1.3422	0.9993	0.0038	1.3708	0.9997	0.0019
1.7	1.1396	0.9838	0.0627	1.3245	0.9967	0.0152	1.3993	0.9990	0.0053	1.4422	0.9996	0.0023	1.4708	0.9998	0.0010
1.8	1.2383	0.9891	0.0442	1.4243	0.9979	0.0098	1.4992	0.9994	0.0032	1.5421	0.9998	0.0013	1.5708	0.9999	0.0005
1.9	1.3374	0.9928	0.0305	1.5241	0.9987	0.0063	1.5992	0.9996	0.0019	1.6421	0.9999	0.0007	1.6707	0.9999	0.0003
2.0	1.4368	0.9953	0.0207	1.6240	0.9992	0.0039	1.6991	0.9998	0.0011	1.7421	0.9999	0.0004	1.7707	1.0000	0.0002
2.1	1.5364	0.9970	0.0137	1.7240	0.9995	0.0024	1.7991	0.9999	0.0007	1.8421	1.0000	0.0002	1.8707	1.0000	0.0001
2.2	1.6362	0.9981	0.0089	1.8239	0.9997	0.0015	1.8991	0.9999	0.0004	1.9421	1.0000	0.0001	1.9707	1.0000	0.0000
2.3	1.7360	0.9989	0.0057	1.9239	0.9998	0.0008	1.9991	1.0000	0.0002	2.0421	1.0000	0.0001	-----	-----	-----
2.4	1.8359	0.9993	0.0036	2.0239	0.9999	0.0005	2.0991	1.0000	0.0001	2.1421	1.0000	0.0000	-----	-----	-----
2.5	1.9359	0.9996	0.0022	2.1239	1.0000	0.0003	2.1991	1.0000	0.0001	-----	-----	-----	-----	-----	-----
2.6	2.0359	0.9998	0.0013	2.2239	1.0000	0.0002	2.2991	1.0000	0.0000	-----	-----	-----	-----	-----	-----
2.7	2.1358	0.9999	0.0008	2.3239	1.0000	0.0001	-----	-----	-----	-----	-----	-----	-----	-----	-----
2.8	2.2358	0.9999	0.0004	2.4239	1.0000	0.0000	-----	-----	-----	-----	-----	-----	-----	-----	-----
2.9	2.3358	1.0000	0.0003	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
3.0	2.4358	1.0000	0.0001	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
3.1	2.5358	1.0000	0.0001	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
3.2	2.6358	1.0000	0.0000	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

TABLE 2

n	α_n	$\zeta''_{n,0}(0)$	$\zeta_{n,1}(\infty)$	$\zeta''_{n,1}(0)$
0	0.5642	1.1284	0.418	1.607
1	0.3761	2.2568	0.138	0.963
2	0.3009	3.0090	0.072	0.756
3	0.2579	3.6108	0.046	0.632
4	0.2293	4.1266	0.033	0.552

TABLE 3

START OF SEPARATION t_A AND DISTANCE TRAVELLED S_A IN THE
CASE OF A CIRCULAR CYLINDER MOVED RECTILINEARLY
OUT OF A STATE OF REST

n	$t_A \left\{ \frac{C_0}{R} \right\}^{\frac{1}{n+1}}$	$\frac{S_A}{R}$
0	0.351	0.351
1	1.082	0.586
2	1.258	0.663
3	1.300	0.714
4	1.302	0.748

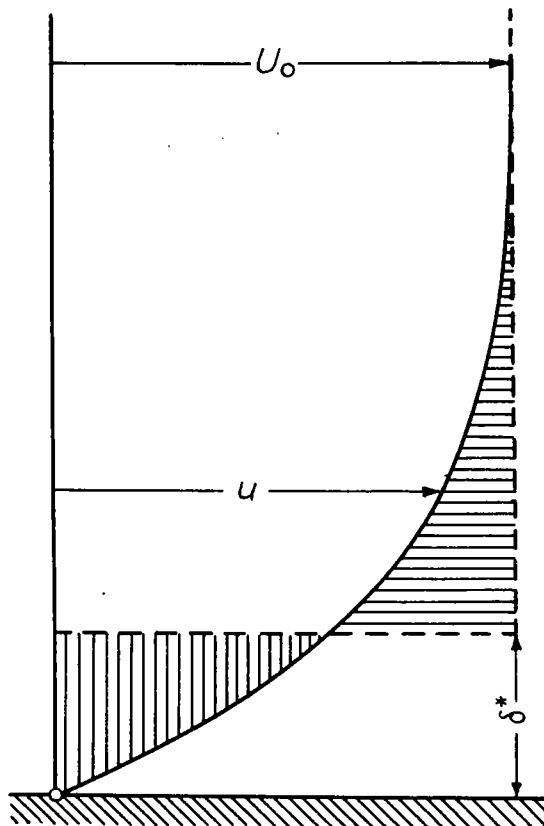


Figure 1.- Regarding definition of δ^* .

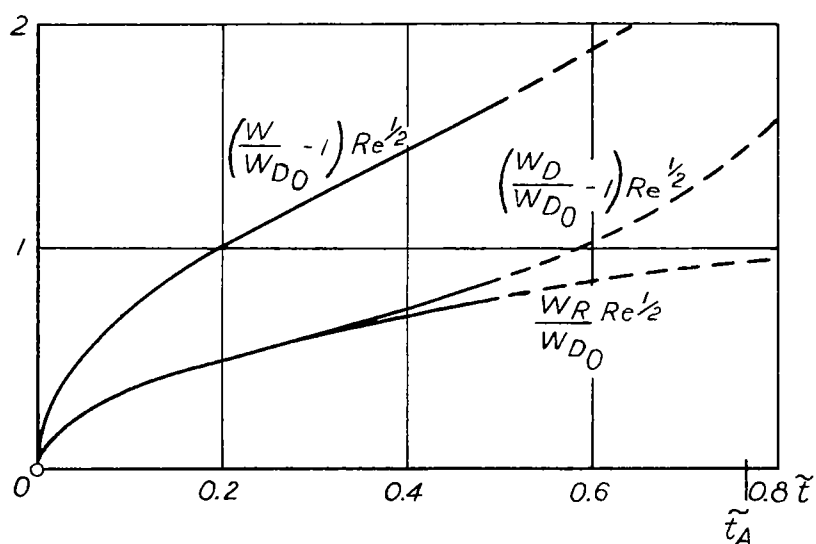


Figure 2.- Pressure drag W_D , friction drag W_R , and total drag $W = W_D + W_R$ of the circular cylinder for very small times after start of the motion for uniform acceleration out of a state of rest.

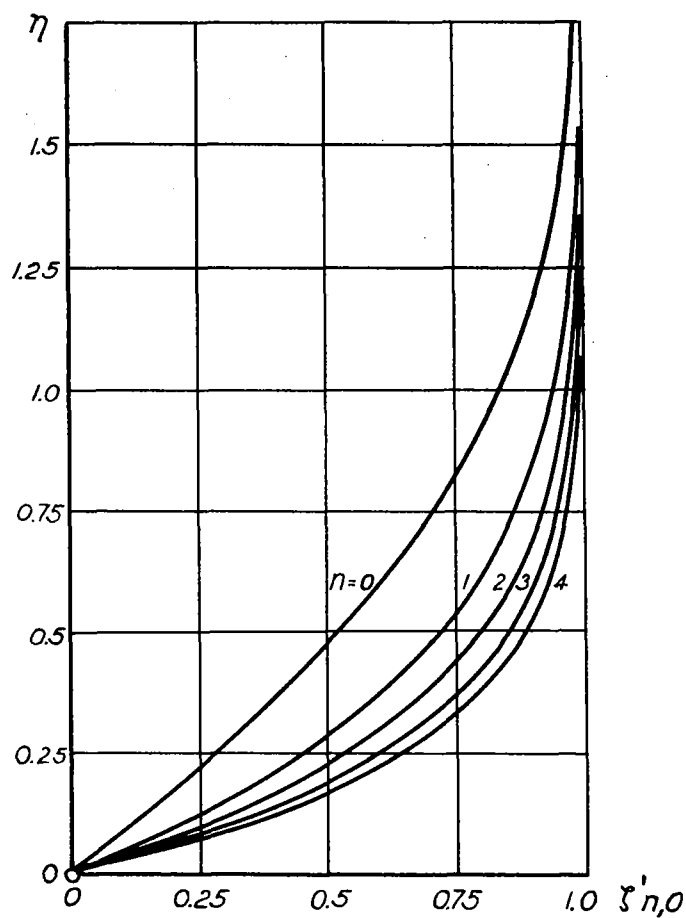


Figure 3.- Course of the functions $\xi_{n,0}(\eta)$ for $n = 0, 1, 2, 3$, and 4 .

NASA Technical Library



3 1176 01441 2002